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Dynamics of the passive scalar in compressible turbulent flow: Large-scale patterns and small-scale fluctuations

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The work analyzes fluctuations of passive scalar and large-scale (mean field) effects in a turbulent compressible fluid flow. It is shown that passive scalar transport can be accompanied by slow diffusion of small-scale inhomogeneous fluctuating structures for large Péclet numbers, $\mathrm{Pe} \gg 1$. The origin of the inhibition of the diffusion of small-scale fluctuations of the passive scalar is associated with compressibility (i.e., $\mathrm{div}\,\mathbf{u} \propto \partial\rho/\partial t \neq 0$) of a surrounding fluid flow. The conditions for the slow diffusion of the passive scalar fluctuations in homogeneous and isotropic turbulent flow are found. It is shown that the magnitude of the fluctuations of the passive scalar generated in the presence of external gradient of the mean mass concentration $\boldsymbol{\nabla} Q$ in compressible fluid flow can be fairly strong: $\sqrt{\langle q^2\rangle} \sim l_0 \ln(\mathrm{Pe})|\boldsymbol{\nabla} Q|$, where l_0 is the characteristic scale of the turbulent velocity field. The characteristic spatial scale of a localization of solutions is of the order of $l_0/\sqrt{\mathrm{Pe}}$. In addition, compressibility in the stratified turbulent inhomogeneous fluid flow [i.e., $\mathrm{div}\,\mathbf{u} = -(\boldsymbol{\nabla}\rho \cdot \mathbf{u})/\rho \neq 0$] results in formation of large-scale structures for large Péclet numbers. The formation of these patterns is caused by the instability of the uniform distribution of the mean passive scalar field whereby an additional nondiffusive component of the flux of passive scalar particles results in a large-scale pattern. The conditions for the excitation of the instability of the mean field are found. Possible environmental applications of these effects are discussed.

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I. INTRODUCTION

The large variety of interesting phenomena related to the passive scalar transport in a random incompressible fluid flow were investigated both theoretically and experimentally (see, e.g., [1–35]). These effects include anomalous turbulent diffusion, intermittency, and fractal structure of a concentration field. Recently the state of the art in the field of passive scalar transport by a turbulent incompressible velocity field and the unified mathematical formulation of the problem were discussed in [33,36]. However, the passive scalar transport by a compressible turbulent flow is a subject of relatively few investigations (see, e.g., [37–39]) and some interesting aspects of this problem were not addressed.

In this study we address some issues of passive scalar transport in compressible turbulent flow of fluid that seem of interest in various phenomena. It will be demonstrated that compressibility of turbulent flow plays an essential role in passive scalar dynamics and causes qualitative changes in the properties of both mean passive scalar field and fluctuations. In particular, we show that the compressibility (i.e., $\mathrm{div}\,\mathbf{u} \propto \partial\rho/\partial t \neq 0$) of a surrounding fluid flow results in a slow diffusion of a small-scale fluctuating component of the passive scalar concentration for large Péclet numbers. The conditions for the slow diffusion of the passive scalar fluctuations in homogeneous and isotropic turbulent flow are found. In addition, the magnitude of the fluctuations of the passive scalar generated in the presence of external gradient of the mean mass concentration $\boldsymbol{\nabla} Q$ in compressible flow of fluid can be fairly strong. On the other hand, passive scalar transport in a stratified turbulent fluid flow [i.e., $\mathrm{div}\,\mathbf{u} = -(\boldsymbol{\nabla}\rho \cdot \mathbf{u})/\rho \neq 0$] is accompanied by formation of large-scale structures. Formation of these patterns is due to instability of the mean passive scalar field in an inhomogeneous turbulent velocity field. The analysis of the instability is performed for large Péclet numbers $\mathrm{Pe} = l_0 u_0/D$ where l_0 is a characteristic length scale of a turbulence velocity field, u_0 is a characteristic value of turbulent velocity $\mathbf{u}$, and D is a coefficient of molecular diffusion.

In order to derive both equations for the second moment of the fluctuating field and for the mean passive scalar field at large Péclet numbers we used a method in which diffusion is described by means of an average over an ensemble of random Wiener trajectories. This approach is similar to the method of Feynman integrals over trajectories and it was successfully applied in quantum mechanics, solid state physics [40], magnetohydrodynamics [20,41,42], and passive scalar transport in incompressible turbulent flow [12,20,28,33]. It is shown here that the equations for the mean passive scalar field for $\mathrm{Pe} \gg 1$ and $\mathrm{Pe} \ll 1$ have the same form.

II. THE GOVERNING EQUATIONS

Evolution of the number density of the passive scalar $n_p(t,\mathbf{r})$ is determined by equation of the convective diffusion:

$$\frac{\partial n_p}{\partial t} + \boldsymbol{\nabla}\cdot(n_p \mathbf{v}) = -\boldsymbol{\nabla}\cdot\mathbf{J}\ , \tag{1}$$

where the diffusive flux $\mathbf{J}$ in the absence of gravity is given by

$$\mathbf{J} = -D\boldsymbol{\nabla} n_p\ ,$$

and D is the coefficient of the molecular diffusion, $\mathbf{v}$ is the velocity of the medium. The velocity $\mathbf{v}$ and the density ρ of the medium satisfy the continuity equation

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0\ . \tag{2}$$

Equations (1),(2) yield an equation for the evolution of the mass concentration of a passive scalar $C = m_p n_p/\rho$:

$$\frac{\partial C}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})C = D\Delta C \tag{3}$$

(see, e.g., [43]). Here it is assumed that a passive scalar consists of small particles with mass m_p.

III. WIENER PROCESS AND EQUATIONS FOR BOTH MEAN FIELD AND THE SECOND MOMENT OF FLUCTUATING PASSIVE SCALAR FIELD

We shall use here the stochastic calculus which was invented in the study of Brownian motion by Einstein and Smoluchowski, and was developed by Wiener, Kac, Feynman, and others into a rigorous mathematical theory (see, e.g., [44,20,33], and references therein). This theory has been successfully applied in quantum mechanics, solid state physics [40], magnetohydrodynamics [20,41,42], and passive scalar transport in incompressible turbulent flow [12,20,28,33]. The main object of the stochastic calculus is a Wiener (Brownian) random process that is defined by the properties

$$M\{\mathbf{w}\} = 0\ ,$$
$$M\{w_i w_j\} = t\delta_{ij}\ ,$$

where M is the mathematical expectation over the Wiener paths. The diffusive motion in this method is described by means of an average over an ensemble of random Wiener trajectories. The problem of solution of Eq. (3) reduces to the analysis of the evolution of field $C(t,\mathbf{r})$ along the Wiener path, $\boldsymbol{\xi}_t$:

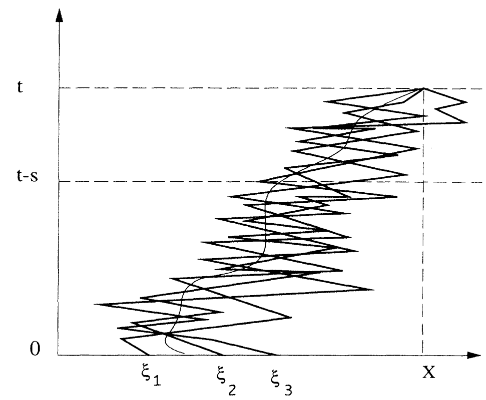

FIG. 1. Random Wiener trajectories. The random trajectories pass through the point $\mathbf{x}$ at time t.

$$\boldsymbol{\xi}_t = \mathbf{x} - \int_0^t \mathbf{v}(t_s, \boldsymbol{\xi}_s)\, ds + (2D)^{1/2}\mathbf{w}(t)\ , \tag{4}$$

where $t_s = t - s$. Equation (4) describes a set of the random trajectories which pass through the point $\mathbf{x}$ at time t (see Fig. 1). Equation (4) is a stochastic integral equation [45]. Since $\mathbf{w}(t)$ is a Wiener process, the initial coordinates of every trajectory $\boldsymbol{\xi}_t$ are random. Without diffusion $[D = 0]$ the Wiener paths coincide with the Lagrangian trajectory and $\boldsymbol{\xi}_t$ is not random.

The solution of Eq. (3) with the initial condition $C(t = 0,\mathbf{x}) = C_0(\mathbf{x})$ is given by

$$C(t,\mathbf{x}) = M\{C_0(\boldsymbol{\xi}_t)\} \tag{5}$$

(see Appendix A). Equation (5) is a particular case of the Feynman-Kac formula (see, e.g., [44,45]).

Now let us derive both equations for the mean passive scalar field and for the second moment of the fluctuating passive scalar component using Eq. (5). The procedure of derivation is outlined in the following.

(i) If the total field C is specified at instant t, then we can determine the total field $C(t+\Delta t)$ at near instant $t+\Delta t$ by means of Eq. (5):

$$C(t+\Delta t,\mathbf{x}) = M\{C(t,\boldsymbol{\xi}_{\Delta t})\}\ ,$$

where

$$\boldsymbol{\xi}_{\Delta t} = \mathbf{x} - \int_0^{\Delta t} \mathbf{v}(t_\sigma,\boldsymbol{\xi}_\sigma)\, d\sigma + (2D)^{1/2}\mathbf{w}(\Delta t)\ ,$$

and $t_\sigma = t + \Delta t - \sigma$.

(ii) Expansion of the function $C(t,\boldsymbol{\xi}_{\Delta t})$ and the velocity $v_m(t_\sigma,\boldsymbol{\xi}_\sigma)$ in Taylor series in the vicinity of the point $\mathbf{x}$ allows us to express the field $C(t+\Delta t,\mathbf{x})$ in terms of the field $C(t,\mathbf{x})$

$$C(t+\Delta t,\mathbf{x}) = M\bigg\{C(t,\mathbf{x}) + \frac{\partial C}{\partial x_m}\bigg[-v_m\Delta t + \frac{1}{2}v_l\frac{\partial v_m}{\partial x_l}(\Delta t)^2 + \sqrt{2D}w_m - \sqrt{2D}\frac{\partial v_m}{\partial x_l}\int_0^{\Delta t} w_l\, ds\bigg] + \frac{1}{2}\frac{\partial^2 C}{\partial x_m \partial x_p}\bigg[v_m v_p(\Delta t)^2 + 2Dw_m w_p + \sqrt{2D}\Delta t(v_m w_p - v_p w_m)\bigg]\bigg\} \tag{6}$$

(see Appendix B). Here we keep terms up to $\geq O[(\Delta t)^2]$ in the expansion.

(iii) In order to determine the mean field Q we average Eq. (6) over the turbulent velocity $\mathbf{u}$, (i.e., $Q = \langle C \rangle$). Note that $\mathbf{v} = \mathbf{V} + \mathbf{u}$, where $\mathbf{V} = \langle \mathbf{v} \rangle$ is the mean velocity and $\mathbf{u}$ is the turbulent component of the velocity. It is important to note that the Wiener random process $\mathbf{w}(t)$ and the turbulent velocity $\mathbf{u}(t,\mathbf{x})$ are independent random processes, and therefore we can change the order of averaging: $\langle M\{f\}\rangle \to M\{\langle f \rangle\}$ (see [20]). On the contrary, the random processes $\mathbf{w}(t)$ and $\mathbf{u}(t, \boldsymbol{\xi}_{\Delta t})$ are correlated. We also assume that the velocities $\mathbf{u}$ in both intervals $(0,t)$ and $(t, t+\Delta t)$ are independent, because we consider the random flow with short time of the renewal. It is assumed also that the velocity $\mathbf{v}$ is constant (time independent) in small intervals $(0,\Delta t); (\Delta t, 2\Delta t); \ldots$, and changes every small time interval Δt. Note that averaging over the Wiener paths corresponds to the averaging over the molecular processes with very small characteristic scales. On the other hand, $\langle f \rangle$ determines the averaging over the turbulent velocity field with scales that are larger than molecular ones.

(iv) Now we calculate

$$\frac{Q(t+\Delta t, \mathbf{x}) - Q(t, \mathbf{x})}{\Delta t} ,$$

and pass to the limit $\Delta t \to 0$. In such a procedure the turbulent velocity field $\mathbf{u}$ with very short time of the renewal tends to a δ-correlated in time random process:

$$\langle u_m(t,\mathbf{x}) u_n(t',\mathbf{y})\rangle = 2\tau_0 \delta(t-t') \langle u_m(\mathbf{x}) u_n(\mathbf{y})\rangle$$

(see, e.g., [20,41,33]). This procedure yields the equation for the mean field Q in the form

$$\frac{\partial Q}{\partial t} + (\mathbf{V}_{\text{eff}} \cdot \boldsymbol{\nabla}) Q = \frac{\partial}{\partial x_p}\left[D_{pm}\frac{\partial Q}{\partial x_m}\right] , \quad (7)$$

where

$$D_{pm} = D\delta_{pm} + \tau_0 \langle u_p u_m \rangle , \quad (8)$$
$$\mathbf{V}_{\text{eff}} = \mathbf{V} + \tau_0 \langle \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{u})\rangle . \quad (9)$$

Note that we use the δ-correlated in time random process to describe the turbulent velocity field only for simplicity. The results remain valid also for a velocity field with a finite correlation time if the statistical characteristics of the passive scalar vary slowly in comparison with the correlation time of the turbulent flow (see, e.g., [46]). We will show in Appendix C that for Pe $\ll 1$ and an arbitrary velocity field the equation for the mean field coincides with Eq. (7).

In the case $(\boldsymbol{\nabla}\cdot\mathbf{v}) = 0$ the effective velocity $\mathbf{V}_{\text{eff}} = \mathbf{0}$ for $\mathbf{V} = 0$, and Eq. (7) coincides with that derived in [20,33]. It is well known that in incompressible turbulent flow with Pe $\gg 1$ molecular diffusion can be neglected in comparison with the turbulent diffusion of the passive scalar. However, the situation is much more complicated in a turbulent compressible flow. In particular, we show that there are conditions for pattern formation in the large-scale distribution of the passive scalar (see Sec. V). The inhomogeneous structures are formed due to an excitation of the instability. The growth rate of the instability depends on the magnitude of $(\boldsymbol{\nabla}\cdot\mathbf{v})$.

(v) Let us calculate the correlator $\varphi(t+\Delta t, \mathbf{x}, \mathbf{y}) = \langle C(t+\Delta t, \mathbf{x}) C(t+\Delta t, \mathbf{y})\rangle$ by means of Eq. (6). The obtained equation allows us to find the function

$$\frac{\varphi(t+\Delta t) - \varphi(t)}{\Delta t} .$$

Passing to the limit $\Delta t \to 0$ yields the equation for the correlation function φ

$$\frac{\partial \varphi}{\partial t} = \hat{L}\varphi ,$$

where

$$\hat{L} = -(\mathbf{V}_{\text{eff}}\cdot\boldsymbol{\nabla})_{\mathbf{x}} - (\mathbf{V}_{\text{eff}}\cdot\boldsymbol{\nabla})_{\mathbf{y}} + [\boldsymbol{\nabla}(\hat{D}\cdot\boldsymbol{\nabla})]_{\mathbf{x}} + [\boldsymbol{\nabla}(\hat{D}\cdot\boldsymbol{\nabla})]_{\mathbf{y}} + 2\tau_0 \langle u_m(\mathbf{x}) u_n(\mathbf{y})\rangle \frac{\partial^2}{\partial x_m \partial y_n} ,$$

and $\hat{D} = D_{mn}$.

(vi) Consider the structure function $\Phi = \varphi - \varphi_0 \equiv \langle q(t,\mathbf{x}) q(t,\mathbf{y})\rangle$, where $q(t,\mathbf{x}) = C(t,\mathbf{x}) - Q(t,\mathbf{x})$, $\varphi_0 = Q(t,\mathbf{x})Q(t,\mathbf{y})$, and $Q(t,\mathbf{x}) = \langle C(t,\mathbf{x})\rangle$ is the mean field. To derive the equation for the structure function Φ we use Eq. (7) for the mean field. The equation for the function Φ is given by

$$\frac{\partial \Phi}{\partial t} = \hat{L}\Phi + I , \quad (10)$$

where

$$I = 2\tau_0 \langle u_m(\mathbf{x}) u_n(\mathbf{y})\rangle \boldsymbol{\nabla}_m Q(\mathbf{x}) \boldsymbol{\nabla}_n Q(\mathbf{y}) . \quad (11)$$

In the case $(\boldsymbol{\nabla}\cdot\mathbf{v}) = 0$ and $\mathbf{V} = 0$ the effective velocity $\mathbf{V}_{\text{eff}} = \mathbf{0}$, and Eq. (10) coincides with corresponding equations that were obtained in [8,20,33]. In the next section we will show that there are certain conditions for slow (inhibited) diffusion of small-scale fluctuations of the passive scalar field. This effect arises due to compressibility $(\boldsymbol{\nabla}\cdot\mathbf{v} \neq 0)$ of turbulent flow.

IV. SMALL-SCALE FLUCTUATIONS OF THE PASSIVE SCALAR FOR LARGE PÉCLET NUMBERS

In this section we will show that a compressibility of surrounding fluid flow results in inhibition of turbulent diffusion of the fluctuations of the passive scalar. The physical mechanism of this effect is quite clear. In incompressible flow at any time the mass of fluid flowing into a small volume exactly equals the mass outflow from this volume. In the limit of infinite Péclet number particles of the passive scalar are frozen into a flow of a surrounding fluid. This means that there is no accumulation of the particles of the passive scalar at any point of the volume. A molecular diffusion results only in a deviation of the particles from their Lagrangian trajectories without accumulation of the matter of the passive scalar.

The situation changes if $\boldsymbol{\nabla}\cdot\mathbf{u} \neq 0$ in a turbulent fluid flow. In this case a mass of fluid flowing into a small volume does not equal a mass outflow from the volume at any instance. Therefore at times smaller than a characteristic time of the turbulent velocity field there is accumulation (or outflow) of the particles of the passive scalar. This accumulation (or outflow) of the particles can strongly reduce turbulent diffusion. At large times the accumulation of the particles of the passive scalar on the average is compensated by outflow of the particles. Note that accumulation and outflow of the particles of the passive scalar in a small control volume are separated in time and molecular diffusion breaks a symmetry between accumulation and outflow (i.e., it breaks a reversibility in time). The latter causes an effective reduction of turbulent diffusion of the fluctuations of the passive scalar concentration.

A. Model of compressible homogeneous and isotropic turbulent velocity field

In this section we study the evolution of the passive scalar fluctuations in a prescribed velocity field $\mathbf{v}$ with $\mathbf{\nabla}\cdot\mathbf{u} \neq 0$, where the velocity of the flow $\mathbf{v} = \mathbf{V}+\mathbf{u}$, $\mathbf{V} = \langle\mathbf{v}\rangle$ is the mean velocity, and $\mathbf{u}$ is the turbulent component of the velocity; the angular brackets mean statistical averaging. This model corresponds to homogeneous and isotropic turbulence with a small, but finite value of $\langle(\mathbf{\nabla}\cdot\mathbf{u})^2\rangle$. In this case the correlation function of the velocity field is given by

$$\langle u_m(\mathbf{x})u_n(\mathbf{y})\rangle = \frac{u_0^2}{3}\bigg[[F(r)+F_c(r)]\delta_{mn} + \frac{r}{2}\frac{dF}{dr}\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right) + r\frac{dF_c}{dr}\frac{r_m r_n}{r^2}\bigg] \tag{12}$$

(see Appendix D), where $F(0) = 1 - F_c(0)$. The function $F_c(r)$ describes the compressible (potential) component whereas $F(r)$ corresponds to vortical part of the turbulence.

B. Slow diffusion of passive scalar fluctuations

We consider here homogeneous and isotropic turbulent flow whereby the mean large-scale inhomogeneous field of the passive scalar cannot be generated (see Sec. V). We assume also that there is no external gradient of the mean concentration. In this case the source $I = 0$ [see Eqs. (10) and (11) for the correlation function Φ]. The function Φ in homogeneous and isotropic turbulence with zero mean field depends only on $r = |\mathbf{x}-\mathbf{y}|$ and t. Therefore Eq. (10) is reduced to

$$\frac{\partial \Phi}{\partial t} = D_* \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + 6\lambda D_T\frac{\partial\Phi}{\partial r} , \tag{13}$$

where

$$\frac{1}{m} = \frac{2}{\mathrm{Pe}} + \frac{2}{3}[1-F-(rF_c)'], \quad D_T = \frac{1}{3}u_0^2\tau_0,$$
$$D_* = \frac{3D_T}{m}, \quad \lambda = -\frac{1}{3}(F-2F_c)' ,$$

$R' = dR/dr$, and we use here the identities

$$\frac{\partial^2\Phi}{\partial r_m \partial r_n} = \frac{\partial^2\Phi}{\partial r^2}\frac{r_m r_n}{r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r}\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right) , \tag{14}$$

$$\frac{\partial\Phi}{\partial x_m} = -\frac{\partial\Phi}{\partial y_m} = \frac{\partial\Phi}{\partial r_m} . \tag{15}$$

The boundary conditions for Eq. (13) are $\Phi(r=0) = \Phi'(r=0) = 1$.

Note that the value

$$D_* = 2u_0^2\tau_0\left[\frac{1}{\mathrm{Pe}} + \frac{1}{3}[1-F-(rF_c)']\right]$$

can be interpreted as a turbulent diffusion coefficient that depends on the scale r. For $r \to 0$ the value D_* tends to $6D_T/\mathrm{Pe} \equiv 2D$, where D is the molecular diffusion coefficient. This means that the turbulent diffusion coefficient tends to the molecular diffusion at very small scales. On the other hand, in large scales (i.e., for $r \gg l_0 = u_0\tau_0$) the functions F, and F_c, and $(rF_c)'$ tend to 0 and $m^{-1} \to 2/3$. Therefore the value D_* recovers the known coefficient of turbulent diffusion in large scales. A similar form of the turbulent diffusion coefficient which gives correct values for both asymptotic cases (very small scales and very large scales) was suggested in [13]. The scale dependent diffusivity occurs also in a case of anisotropic turbulence as was proved rigorously for a random shear flow in [33].

Now we introduce a function

$$\psi = r\Phi(t,r)\exp\left[\int_0^r \chi(x)\,dx\right] \tag{16}$$

and use Eq. (13). It results in an equation for the function $\psi(r)$ in nondimensional form

$$\frac{\partial\psi}{\partial t} = \frac{1}{m}\frac{\partial^2\psi}{\partial r^2} - U(r)\psi , \tag{17}$$

where the function $U(r)$ is given by

$$U = \lambda\left(\frac{2}{r} + \chi + \frac{\chi'}{\chi}\right), \quad \chi = \lambda m ,$$

and time t is measured in units of τ_0, distance r is measured in units of $l_0 = u_0\tau_0$.

We seek a solution to the equation for ψ of the form

$$\psi(t,r) = \exp(2\gamma t)\Psi(r). \tag{18}$$

Thus Eq. (17) is reduced to the eigenvalue problem:

$$\frac{1}{m(r)}\frac{d^2\Psi}{dr^2} - [2\gamma + U(r)]\Psi = 0 . \tag{19}$$

In the analysis we use a quantum mechanics analogy (see, e.g., [47]) whereby Eq. (19) is a one-dimensional Schrödinger equation with a variable mass. In the limit of large Péclet number the formulation is amenable to treatment by an asymptotic method (see, e.g., [20,42]). A particular form of the potential $U(r)$ and the mass $m(r)$ depend on the functions $F(r)$, $F_c(r)$, and the Péclet number. For instance, we may choose these functions in the form

$$F(r) = (1-\varepsilon)\exp(-r^2), \quad F_c(r) = \varepsilon \exp(-\alpha r^2) . \quad (20)$$

Here ε determines the energy of the potential component of the turbulent velocity and $\alpha^{-1/2}$ is the characteristic scale of the potential component of flow.

Using the formulas derived in Appendix D it can be shown that

$$\langle(\nabla\cdot\mathbf{u})^2\rangle \equiv -\frac{\partial^2}{\partial r_m \partial r_n}\langle u_m u_n\rangle\bigg|_{r\to 0} = -\frac{8}{3}u_0^2\left(\frac{F_c'}{r}+\frac{7}{8}F_c''\right)_{r\to 0} .$$

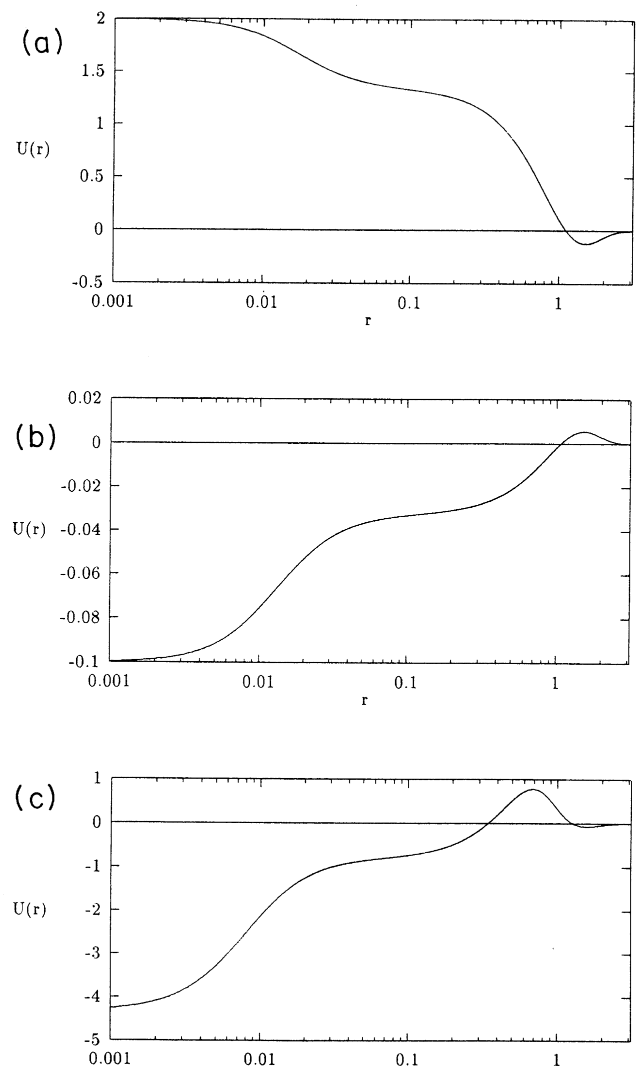

FIG. 2. Distribution of potential $U(r)$ for Pe = 10^4 and different values of ε and α: (a) $\varepsilon = 0$; (b) $\varepsilon = 0.35$, $\alpha = 1$; (c) $\varepsilon = 0.35$, $\alpha = 4$.

For $r \ll 1$ the function $F_c = \varepsilon(1-\alpha r^2)$ and

$$\langle(\nabla\cdot\mathbf{u})^2\rangle = 10\varepsilon\alpha u_0^2 .$$

The potential $U(r)$ and mass $m(r)$ in the form (20) are plotted in Figs. 2 and 3. Note that the distribution of mass weakly depends on ε and α. A region with positive potential determines the diffusion of the fluctuating component of the passive scalar concentration. Note that a strong decrease of the mass at small r results in a strong localization of the solution.

Usually the Schrödinger equation with such complicated potential and mass can be solved only numerically. However, availability of a small parameter $(\ln \mathrm{Pe})^{-1} \ll 1$ allows us to obtain an asymptotic solution of this equation. It is seen from the distribution of the potential $U(r)$ (Fig. 2) that there are three characteristic regions, in which mass $m(r)$, potential $U(r)$, and Eq. (19) can be reduced to the following form.

Region I. $0 < r \ll r_1$:

$$\frac{1}{m(r)} = \frac{2}{\mathrm{Pe}}[1+(\beta_m \mathrm{Pe})r^2] ,$$

$$U(r) = 2\beta\left(1-\frac{(\beta_U \mathrm{Pe})r^2}{1+(\beta_m \mathrm{Pe})r^2}\right) ,$$

$$(1+X^2)\frac{d^2\Psi(X)}{dX^2} - \frac{1}{\beta_m}\left[\gamma+\beta - \frac{\beta\beta_U}{\beta_m}\left(1-\frac{1}{1+X^2}\right)\right]\Psi(X) = 0 , \quad (21)$$

where $X = (\beta_m \mathrm{Pe})^{1/2} r$, and

$$\beta = (1-\varepsilon)(1-2\sigma), \quad -2\alpha \le \beta \le 1 ,$$

$$\beta_m = \frac{1}{3}(1-\varepsilon)(1+3\sigma), \quad \frac{1}{3} \le \beta_m \le \alpha ,$$

$$\beta_U = \frac{1}{9}(1-\varepsilon)(1+8\sigma), \quad \frac{1}{9} \le \beta_U \le \frac{8}{9}\alpha ,$$

$$\sigma = \frac{\varepsilon\alpha}{1-\varepsilon}, \quad 1-\varepsilon = \frac{\alpha}{\alpha+\sigma} .$$

Region II. $r_1 < r < r_2$:

$$\frac{1}{m} \sim \frac{2}{3}, \quad U \sim U_0 ,$$

$$\frac{d^2\Psi(r)}{dr^2} - \frac{3}{2}(U_0 - 2|\gamma|)\Psi = 0 . \quad (22)$$

Region III. $r > r_2$:

$$\frac{1}{m} \sim \frac{2}{3}, \quad U \sim 0 ,$$

$$\frac{d^2\Psi(r)}{dr^2} + 3|\gamma|\Psi = 0 . \quad (23)$$

We use here the expansion

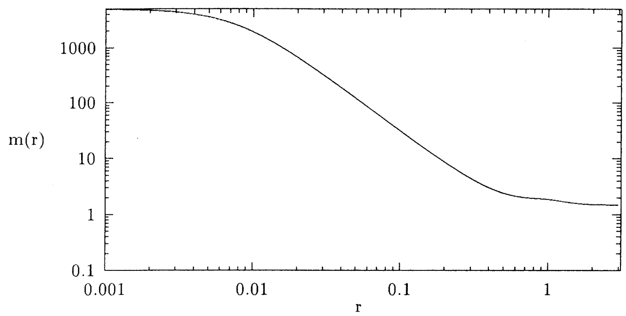

FIG. 3. Distribution of mass $m(r)$ for Pe $= 10^4$ and $\varepsilon = 0.35$, $\alpha = 4$.

$$F(r) \simeq (1-\varepsilon)(1-r^2), \quad F_c(r) \simeq \varepsilon(1-\alpha r^2) \quad \text{for } r \ll 1,$$

$$F(r) \to 0, \quad F_c(r) \to 0 \quad \text{for } r \gg 1 \ ,$$

and consider only the case with $\gamma < 0$ (see below) which corresponds to diffusion of the fluctuating component of the passive scalar field [see Eq. (18)].

First we study the region I. We seek a solution of Eq. (21) in the form

$$\Psi(X) = (1+X^2)^{\frac{1}{2}} \mathrm{Re}[T(iX)] \ ,$$

where $\mathrm{Re}[T(iX)]$ is a real part of the complex function $T(Z)$ that is determined by the Legendre equation:

$$(1-Z^2)T''(Z) - 2ZT'(Z) + \left[\nu(\nu+1) - \frac{\mu^2}{1-Z^2}\right]T(Z) = 0 \ , \quad (24)$$

and

$$\mu^2 = 1 - \beta\frac{\beta_U}{\beta_m^2} \ ,$$

$$\gamma = \beta_m \nu(\nu+1) - \frac{\beta}{\beta_m}\left(1 - \frac{\beta_U}{\beta_m}\right) \ , \quad (25)$$

where $Z = iX$, and $1 \le \mu \le 5/3$ for $\sigma \ge 1/2$. The correlation function Φ at $r = 0$ is positive, i.e.,

$$\Phi(t, r=0) \equiv \langle q^2(t) \rangle = 1 \ . \quad (26)$$

This means that γ should be real. Indeed, it follows from Eqs. (16) and (18) that

$$\Phi(t, r=0) = \mathrm{Re}\left[\left(\frac{\Psi(r)}{r}\right)_{r\to 0} \exp(2\gamma t)\right] .$$

If γ is a complex number, the function $\Phi(t, r=0)$ can be negative in contradiction with the definition of the correlation function. The parameter $\nu = \nu_R + i\nu_I$ is a complex number. It follows from (25) that $\nu_R = -1/2$. Using the explicit form of β, β_m, β_U, ν_R we rewrite the damping rate γ in the form

$$\gamma = -\frac{1}{12(1+3\sigma)}\left(\frac{\alpha}{\alpha+\sigma}\right)[(\sigma-3)^2 + 4\nu_I^2(1+3\sigma)^2] \ , \quad (27)$$

where unknown parameter ν_I will be found below. The general solution of Eq. (24) is given by

$$T(Z) = A_1 P_\nu^\mu(Z) + A_2 Q_\nu^\mu(Z) \ ,$$

where $P_\nu^\mu(Z)$ and $Q_\nu^\mu(Z)$ are the Legendre functions (see, e.g., [48]) with imaginary argument $Z = iX$ and, generally, complex ν. Unknown coefficients A_1 and A_2 can be determined from the boundary condition

$$\Psi(r=0) = 0 \ , \quad (28)$$

and the condition of the normalization (26) for the function Φ. The condition (26) can be rewritten after taking into account Eqs. (16) and (18)

$$\left(\frac{d\Psi}{dr}\right)_{r=0} = 1 \ . \quad (29)$$

Conditions (28) and (29) yield

$$A_1 = -\frac{\pi}{2\sin[\pi(1+2\mu)/4]} \exp\left[\frac{i\pi}{4}(6\mu - 1)\right] A_2 \ ,$$

$$A_2 = \frac{f_1(\mu)}{\sqrt{\beta_m \mathrm{Pe}}} \exp\left[-\frac{i\pi}{2}(3\mu+1)\right] \ ,$$

$$f_1(\mu) = 2^{1-\mu}\pi^{-3/2}\cos(\pi\mu)\left[\Gamma\left(\frac{1-2\mu}{4}\right)\right]^2 .$$

Here we assume that $\nu_I \ll 1$. We will show below that $\nu_I \ll 1$ when $\ln(\mathrm{Pe}) \gg 1$. For the calculation of A_1 and A_2 we use an identity for the gamma function: $\Gamma(Z)\Gamma(1-Z) = \pi/\sin(\pi Z)$ (see, e.g., [48]). To obtain a solution in the vicinity $r \sim r_1$ we use asymptotic formulas for the Legendre functions at $|Z| \simeq r\sqrt{\mathrm{Pe}} \gg 1$. The result is given by

$$\Psi = A r^{1/2} \sin[\nu_I \ln(r\sqrt{\beta_m \mathrm{Pe}})] \ ,$$

where

$$A = \frac{f_1(\mu) f_2(\mu)}{\nu_I(\beta_m \mathrm{Pe})^{1/4}} \ ,$$

$$f_2(\mu) = 2^{-3/2}(\pi + \sqrt{\pi})\left\{\sin\left[\frac{\pi}{4}(1+2\mu)\right] \times\Gamma\left(\frac{1-2\mu}{2}\right)\right\}^{-1} .$$

Here we use the identity

$$Z^{i\nu_I} = [\cos(\nu_I \ln|Z|) + i\sin(\nu_I \ln|Z|)]\exp[-\nu_I \arg(Z)] \ .$$

The solution in region II (i.e., $r_1 < r < r_2$) is given by

$$\Psi = A_3 \exp[\kappa(r - r_2)] + A_4 \exp[-\kappa(r - r_2)] \ ,$$

where $\kappa = [3(U_0 - 2|\gamma|)/2]^{1/2}$.

The solution in region III (i.e., $r > r_2$) is given by

$$\Psi = A_5 \cos[\kappa_1(r - r_2) + \phi] ,$$

where $\kappa_1 = \sqrt{3|\gamma|}$. Unknown coefficients A_3, A_4, A_5, and parameter ν_I can be found by means of matching the function $\Psi(r)$ and its first derivative $d\Psi(r)/dr$ at the points r_1 and r_2. The result is given by

$$\nu_I = \frac{2\pi p}{\ln(\mathrm{Pe})}, \quad p = \pm 1, \pm 2, \dots , \tag{30}$$

and

$$A_3 = \frac{A\nu_I}{\kappa} \exp(\kappa d) \left[1 - \frac{B-1}{B+1} \exp(2\kappa d)\right]^{-1} ,$$
$$A_4 = \frac{B-1}{B+1} A_3, \quad B = -\left(\frac{U_0}{2|\gamma|} - 1\right)^{\frac{1}{2}} \cot\phi ,$$
$$A_5 = \frac{A_3 + A_4}{\cos\phi} ,$$

where $d = r_2 - r_1$. Substitution of (30) into Eq. (27) yields the damping rate of γ of the fluctuating component of the passive scalar field

$$\gamma = -\frac{1}{12(1+3\sigma)} \left(\frac{\alpha}{\alpha+\sigma}\right) \times \left[(\sigma - 3)^2 + \frac{4\pi^2 p^2}{\ln^2(\mathrm{Pe})}(1+3\sigma)^2\right] .$$

The obtained solution is valid for $|\gamma| < U_0/2$. The maximum of the potential U_0 can be much less than unity for a wide range of parameters. The damping rate γ is measured in units of $\tau_0^{-1} = u_0/l_0$. Therefore the characteristic diffusion time can be much longer than the turnover time of the turbulent eddies τ_0. When $\varepsilon = 0$ (i.e., $\mathrm{div}\,\mathbf{u} = 0$) there is only a solution with $|\gamma| > U_0/2 = 1$. This case corresponds to turbulent diffusion. On the other hand, for $\varepsilon > 0$ (when $\mathrm{div}\,\mathbf{u} \neq 0$) there are conditions for very small $|\gamma|$. For instance, $|\gamma| \sim 1/\ln^2(\mathrm{Pe}) \ll 1$ when $\sigma \approx 3$. The latter inequality corresponds to slow diffusion, i.e., compressibility results in fairly strong reduction of the turbulent diffusion of passive scalar fluctuations.

The characteristic scale of the function Ψ in the region I is $l_* \sim (\Psi'/\Psi)^{-1} \sim l_0 \mathrm{Pe}^{-1/2}$. This means that the characteristic scale of the correlation function Φ is $l_* \sim l_0 \mathrm{Pe}^{-1/2}$. When $\varepsilon = 0$ (i.e., $\mathrm{div}\,\mathbf{u} = 0$) the characteristic time of diffusion in this scale is $\tau = l_*^2/D \sim \tau_0$. On the other hand, when $\mathrm{div}\,\mathbf{u} \neq 0$, the diffusion in the scales $l < l_*$ is substantially slower, i.e., the diffusion time is much longer than τ_0. Thus compressibility discourages turbulent diffusion. Note that the main properties of the obtained solution are preserved even in the case $|\gamma| \geq U_0/2$, when $U_0 \ll 1$. However, in this case the solution has a continuous spectrum.

Here we have not considered a solution of the equation for Ψ with positive γ [negative region of the potential $U(r)$]. This solution yields the function Φ, that cannot be a correlation function. Indeed, the main maximum of the function Φ for the case $\gamma > 0$ is located at the region $r > 0$. This fact follows from the analysis of Eq. (13) in the region $r \to 0$. However, according to the definition of a correlation function the main maximum must be located exactly at $r = 0$.

C. Generation of fluctuations of the passive scalar

In the preceding subsection we considered a case where the external gradient of mean mass concentration $\nabla Q = 0$. This means that the source $I = 0$ in Eq. (10) for the correlation function Φ. When $\nabla Q = \mathrm{const} \neq 0$ the nonzero source I results in generation of fluctuations of the passive scalar caused by tangling of external gradient of mean mass concentration by the turbulent velocity field. Here we study this effect in detail. Substituting the correlation function of the velocity field (12) into Eq. (11) for the source I we obtain

$$I = l_0^2(\nabla Q)^2[\lambda_1(r) + r\lambda(r)\cos^2\Theta] ,$$

where the source I is written in dimensionless form, Θ is the angle between the vectors $\mathbf{r}$ and ∇Q, and

$$\lambda_1(r) = \frac{2}{3}\left[F(r) + F_c(r) + \frac{1}{2} r F'(r)\right] .$$

We consider here only spherically symmetric solutions of Eq. (10). In this case Eq. (10) is reduced to

$$\frac{\partial \Phi}{\partial t} = \frac{1}{m(r) r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) + 2\lambda \frac{\partial \Phi}{\partial r} + I_*(r) , \tag{31}$$

where

$$I_*(r) = l_0^2(\nabla Q)^2\left[\lambda_1(r) + \frac{1}{3} r\lambda(r)\right] .$$

Using substitution (16) in Eq. (31) we obtain the one-dimensional Schrödinger equation with a variable mass

$$\frac{\partial \psi}{\partial t} = \frac{1}{m} \frac{\partial^2 \psi}{\partial r^2} - U(r)\psi + r I_*(r) \exp\left[\int_0^r \chi(x)\, dx\right] . \tag{32}$$

We seek a solution of Eq. (32) in the form

$$\psi(t, r) = \sum_{p=1}^{\infty} b_p(t) \Psi_p(r) , \tag{33}$$

where $\Psi_p(r)$ are the eigenfunctions determined by Eq. (19). The condition of the orthogonality for the eigenfunctions is given by

$$\int_0^\infty m(r) \Psi_p(r) \Psi_l(r)\, dr = N_0 \delta_{pl} , \tag{34}$$

where $N_0 \simeq \mathrm{Pe}^{-1/2}/6$. Here we take into account that the main contribution to the integral (34) is from the region $r < \mathrm{Pe}^{-1/2}$. In this region the mass $m(r) \simeq \mathrm{Pe}/2$, and the mass drastically drops for $r \geq \mathrm{Pe}^{-1/2}$ up to the value $m(r) \simeq 3/2$. Substitute (33) into Eq. (32), multiply the obtained equation by $\Psi_l(r)$, and integrate over r. The result is given by

$$\frac{d}{dt}b_p - \gamma_p b_p = \frac{2}{3}l_0^2(\boldsymbol{\nabla}Q)^2 , \tag{35}$$

where we use Eq. (34) and take into account that

$$\int_0^\infty m(r) r I_*(r) \exp\left(\int_0^r \chi(x)\,dx\right)\Psi_p(r)\,dr \simeq \frac{1}{9}l_0^2(\boldsymbol{\nabla}Q)^2 \mathrm{Pe}^{-1/2} .$$

The solution of Eq. (35) with initial condition $b_p(t=0)=0$ is given by

$$b_p(t) = \frac{2}{3}\frac{l_0^2}{|\gamma_p|}(\boldsymbol{\nabla}Q)^2[1-\exp(-|\gamma_p|t)] . \tag{36}$$

Now we take into account Eqs. (16), (33), and (36) to obtain the solution for the correlation function $\Phi(t,r)$:

$$\Phi(t,r) = \frac{2}{3}\left(\frac{l_0}{L}\right)^2(\delta Q)^2\sum_{p=1}^{\infty}\frac{\Phi_p(r)}{|\gamma_p|}[1-\exp(-|\gamma_p|t)] , \tag{37}$$

where we used an estimate $(\boldsymbol{\nabla}Q)^2 \sim (\delta Q)^2/L^2$. The main contribution to the correlation function $\Phi(t,r)$ for $t \gg |\gamma_p|^{-1}$ in (37) is due to the mode with minimum damping rate $|\gamma_p|$, i.e., for $p=1$. Therefore, a level of fluctuations of the passive scalar in a steady state is given by

$$\sqrt{\langle q^2(r=0)\rangle} \sim \frac{1}{\sqrt{|\gamma_1|}}\left(\frac{l_0}{L}\right)(\delta Q) .$$

Note that in incompressible flow $|\gamma| \sim 1$ and the level of the fluctuations of the passive scalar $\sqrt{\langle q^2\rangle} \ll \delta Q$, since $l_0 \ll L$. However, in compressible flow the damping rate $|\gamma_1|$ can be small. For instance, when $\sigma \to 3$ the damping rate $|\gamma_1| \sim [\ln(\mathrm{Pe})]^{-2} \ll 1$. This means that the level of the fluctuations of the passive scalar generated in the presence of external gradient of the mean mass concentration in compressible flow can be fairly strong,

$$\sqrt{\langle q^2\rangle} \sim \ln(\mathrm{Pe})\left(\frac{l_0}{L}\right)(\delta Q) .$$

V. DYNAMICS OF THE MEAN PASSIVE SCALAR FIELD FOR LARGE PÉCLET NUMBER

A. Effect of gravitation

Now let us take into account the gravity field. In this case additional terms appear in Eq. (3). We consider a case of the heavy passive scalar (i.e., a passive scalar consisting of particles with mass $m_p \gg m_f$, m_f is a molecular weight of a surrounding fluid). In this case the number density evolution is governed by the equation of convective diffusion (1) except for the different formula for diffusive flux $\mathbf{J}$ written in the form suggested by Smoluchowski,

$$\mathbf{J} = -D\left(\boldsymbol{\nabla}n_p - \frac{m_p\mathbf{g}}{\kappa T}n_p\right) \tag{38}$$

(see, e.g., [49]), where κ is a Boltzmann constant, $\mathbf{g}$ is gravity acceleration, and T is a temperature of a surrounding fluid. The second term in the flux $\mathbf{J}$ describes a sedimentation of the passive scalar particles in a gravity field with a velocity $\mathbf{v}_s = D(m_p\mathbf{g}/\kappa T)$. This means that the velocity of the particles of the passive scalar is given by $\mathbf{v}_p = \mathbf{v} + \mathbf{v}_s$. The stationary solution of Eqs. (1), (2), and (38) in the absence of fluid flow is determined by the equation $\boldsymbol{\nabla}n_p - (m_p\mathbf{g}/\kappa T)n_p = 0$. Solution of the latter equation is given by a barometric distribution with a height length scale for particles $\Lambda_p = \kappa T/m_p g \ll |\boldsymbol{\nabla}\rho/\rho|^{-1}$.

For example, we consider a Brownian approximation for diffusivity of heavy particles. In this case the coefficient of diffusion is given by

$$D = \frac{\kappa T}{6\pi a_* \rho\nu}$$

(see, e.g., [43]), where a_* is the radius of a Brownian particles, ν is the kinematic viscosity of a surrounding fluid. Equations (1), (2), and (38) yield the equation for the evolution of a mass concentration of a passive scalar:

$$\frac{\partial C}{\partial t} + (\mathbf{v}+\mathbf{v}_s)\cdot\boldsymbol{\nabla}C = D\Delta C . \tag{39}$$

Hereafter we neglect small terms $\propto DC/\Lambda^2$, $\propto |\boldsymbol{\nabla}D\cdot\boldsymbol{\nabla}C|$, and $\propto |C\boldsymbol{\nabla}D|/\Lambda$ in Eq. (39) for the case $\mathrm{Pe} \gg 1$, where $\Lambda = |\boldsymbol{\nabla}\rho/\rho|^{-1}$. We also neglect a small component of velocity $\propto D/\Lambda \ll |\mathbf{v}_s|$ in Eq. (39). Therefore Eq. (39) written in a frame moving with a velocity $\mathbf{v}_s$ is reduced to the form (3).

B. Model of an inhomogeneous turbulent velocity field in a stratified medium

In this section we study the evolution of the mean passive scalar field in a prescribed velocity field $\mathbf{v}$. In this sense we consider a kinematic problem. The velocity of the flow $\mathbf{v} = \mathbf{V} + \mathbf{u}$, where $\mathbf{V} = \langle\mathbf{v}\rangle$ is the mean velocity and $\mathbf{u}$ is the turbulent component of the velocity; the angular brackets mean statistical averaging. We consider a model of the turbulent velocity field $\mathbf{u}$ with $\boldsymbol{\nabla}\cdot\mathbf{u} \neq 0$ that corresponds to the flow in a stratified medium. In this case the correlation function of the velocity field $\langle u_m(\mathbf{x})u_n(\mathbf{y})\rangle$ is given by

$$\langle u_m(\mathbf{y}) u_n(\mathbf{x}) \rangle = B_* \Bigg[F_0(r, \mathbf{R}) \delta_{mn} + \frac{r}{2} \frac{\partial F_0}{\partial r} \left(\delta_{mn} - \frac{r_m r_n}{r^2} \right) - \frac{1}{2} F_0(r, \mathbf{R}) (r_m \lambda_n - r_n \lambda_m) + \lambda^2 A(r, \mathbf{R}) \left(\delta_{mn} - \frac{\lambda_m \lambda_n}{\lambda^2} \right) - \frac{1}{4} \left(r_m \frac{\partial}{\partial R_n} - r_n \frac{\partial}{\partial R_m} \right) F_0(r, \mathbf{R}) + \frac{1}{4} \delta_{mn} \left(\frac{\partial^2}{\partial R^2} - 4 \lambda_p \frac{\partial}{\partial R_p} \right) A(r, \mathbf{R}) - \frac{1}{4} \frac{\partial}{\partial R_m} \frac{\partial}{\partial R_n} A(r, \mathbf{R}) + \frac{1}{2} \left(\lambda_m \frac{\partial}{\partial R_n} + \lambda_n \frac{\partial}{\partial R_m} \right) A(r, \mathbf{R}) \Bigg] \quad (40)$$

(see Appendix E), where

$$F_0(r, \mathbf{R}) = -\frac{2}{r} \frac{\partial A}{\partial r}, \quad \mathbf{r} = \mathbf{x} - \mathbf{y}, \quad \mathbf{R} = (\mathbf{x} + \mathbf{y})/2 ,$$

$\boldsymbol{\lambda}$ determines the inhomogeneity of the density of the surrounding flow, i.e., $\boldsymbol{\lambda} = -\boldsymbol{\nabla} \rho / \rho$. Here the functions $F_0(r, \mathbf{R})$ and $A(r, \mathbf{R})$ are assumed to be independent of the direction of the vector $\mathbf{r}$. The turbulent velocity field which is determined by Eq. (40) satisfies the continuity equation $\boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$.

In the case where the vector $\partial F_0 / \partial \mathbf{R}$ as well as $\partial A / \partial \mathbf{R}$ are directed along $\boldsymbol{\lambda}$ and for $\mathbf{x} = \mathbf{y} = \mathbf{z}$ the tensor (40) is given by

$$\langle u_m(\mathbf{z}) u_n(\mathbf{z}) \rangle = \frac{\langle u^2 \rangle}{3} \left(1 + \frac{l_0^2 \lambda^2}{6} \alpha(\mathbf{z}) \right)^{-1} \left[\delta_{mn} + \frac{l_0^2 \lambda^2}{4} \alpha(\mathbf{z}) \left(\delta_{mn} - \frac{\lambda_m \lambda_n}{\lambda^2} \right) \right] , \quad (41)$$

where l_0 is the length scale of the energy containing eddies and

$$\alpha(\mathbf{z}) = 1 + \frac{1}{4\lambda^2} \frac{F_0''}{F_0} - \frac{1}{4\lambda} \frac{F_0'}{F_0} ,$$

where $F_0'(z) = dF_0/dz$, and vector $\boldsymbol{\lambda}$ is directed along the axis z, and we use that

$$F_0(\mathbf{R}) = \frac{4}{l_0^2} A(\mathbf{R})$$

(see Appendix E).

C. Properties of the mean field equation

Evolution of the mean field Q is determined by the equation

$$\frac{\partial Q}{\partial t} + (\mathbf{V}_{\text{eff}} \cdot \boldsymbol{\nabla}) Q = \frac{\partial}{\partial x_p} \left[D_{pm} \frac{\partial Q}{\partial x_m} \right] . \quad (42)$$

In incompressible flow Eq. (42) is reduced to

$$\frac{\partial Q}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla}) Q = \frac{\partial}{\partial x_m} \left[D_{mp} \frac{\partial Q}{\partial x_p} \right] .$$

This is a well known equation which describes the turbulent diffusion. This means that for large Pe the turbulence enhances diffusion in comparison with the molecular diffusion (see, e.g., [20,33]). Therefore in incompressible flow any initial inhomogeneous distribution of the passive scalar is transformed into a homogeneous distribution due to the turbulent diffusion.

Now we study the effect of compressibility on the evolution of the passive scalar. Note that Eq. (42) can be rewritten in the form of a conservation law for the total number of particles

$$\frac{\partial N}{\partial t} + \boldsymbol{\nabla} \cdot \left((\mathbf{V} + \mathbf{v}_s) N + \tau_0 \frac{\boldsymbol{\nabla}_n \rho}{\rho} \langle u_n \mathbf{u} \rangle N - \hat{D} \boldsymbol{\nabla} N \right) = 0 , \quad (43)$$

where $N = \langle n_p \rangle$ is the mean number density of particles, $\hat{D} = D_{mn}$.

Now let us multiply Eq. (42) by Q and after simple manipulations we obtain

$$\frac{\partial Q^2}{\partial t} + (\boldsymbol{\nabla} \cdot \mathbf{S}) = Q^2 (\boldsymbol{\nabla} \cdot \mathbf{V}_{\text{eff}}) - I_D ,$$

where

$$\mathbf{S}_m = Q^2 (\mathbf{V}_{\text{eff}})_m - D_{mn} \boldsymbol{\nabla}_n Q^2,$$

$$I_D = 2 D_{mn} (\boldsymbol{\nabla}_m Q)(\boldsymbol{\nabla}_n Q) .$$

If the field Q is homogeneous, $I_D = 0$, $(\boldsymbol{\nabla} \cdot \mathbf{S}) = Q^2 (\boldsymbol{\nabla} \cdot \mathbf{V}_{\text{eff}})$ and $Q^2 = \text{const}$. However, if the initial distribution of the passive scalar Q is inhomogeneous and $(\boldsymbol{\nabla} \cdot \mathbf{V}_{\text{eff}}) > 0$ there is a condition for excitation of the instability that results in pattern formation. This means that any inhomogeneous perturbation of the initially homogeneous spatial distribution of passive scalar particles will grow, i.e.,

$$\frac{\partial}{\partial t} \int Q^2 \, d^3 r > 0.$$

The latter equation implies that in a stratified turbulent flow of fluid when $\boldsymbol{\nabla} \rho \neq \mathbf{0}$ and $\partial \rho / \partial t = 0$ the value

$$\frac{\partial}{\partial t} \int N^2 \, d^3 r > 0 ,$$

where $Q = N \rho$. Therefore the value $\int N^2 \, d^3 r$ grows, whereas the total number of particles $\int N \, d^3 r$ is conserved [see Eq. (43)]. This means that there is a spatial redistribution of particles so that regions with increased concentration of passive scalar particles are separated from those with decreased concentration of particles.

The reason for $(\nabla \cdot \mathbf{V}_{\rm eff}) \neq 0$ is either the compressibility of the mean flow

$$(\nabla \cdot \mathbf{V}) \neq 0 ,$$

or the compressibility of a turbulent flow

$$[\nabla \cdot \langle \mathbf{u}(\nabla \cdot \mathbf{u})\rangle] \neq 0,$$

or both of them. When $(\nabla \cdot \mathbf{V}) > 0$ the surrounding fluid flows out of the control volume. However, due to molecular and turbulent diffusion the particles of the passive scalar do not follow the trajectories of liquid particles. The latter leads to increase of the mass concentration Q of a passive scalar, i.e., $\int Q\, d^3r$ in the control volume grows. On the other hand, the number density n_p changes slowly in comparison with variation of the density ρ. Thus when the divergence of the mean flow $(\nabla \cdot \mathbf{V}) > 0$, the effect is quite trivial. Note also that the case of a stationary mean flow with $(\nabla \cdot \mathbf{V}) \neq 0$ implies either continuous accumulation of the mass of the surrounding fluid at any point of volume, or a constant flow of mass of the fluid from infinity. In this case pressure at this point of a control volume increases, and it will stop the continuous accumulation of the surrounding fluid at this point. Therefore the stationary mean flow with $(\nabla \cdot \mathbf{V}) \neq 0$ is hardly possible.

The situation is different when $(\nabla \cdot \mathbf{V}) = 0$ but $\nabla \cdot \langle \mathbf{u}(\nabla \cdot \mathbf{u})\rangle \neq 0$. In this case the density ρ of a surrounding fluid does not change, but still $\int Q^2\, d^3r$ grows. This effect arises due to the inhomogeneous distribution of a number density of a passive scalar in the turbulent compressible flow.

D. Large-scale instability

Consider the case when $\nabla \cdot \langle \mathbf{u}(\nabla \cdot \mathbf{u})\rangle \neq 0$. Assume for simplicity the turbulent flow with zero mean velocity. Now let us study in detail the instability of the large-scale distribution of the passive scalar in the small-scale turbulent stratified flow. The stratification of the turbulent flow can be caused, for instance, by the inhomogeneous distribution of the density ρ. In the case $\lambda^2 l_0^2/6 \ll 1$ and $l_0^2 F_0''/F_0 < 1$ the expressions for D_{mn} and $\mathbf{V}_{\rm eff}$ are given by

$$D_{mn} = D\delta_{mn} + D_T F_0(\mathbf{z})\delta_{mn} , \tag{44}$$

$$\mathbf{V}_{\rm eff} = D_T F_0(\mathbf{z})\boldsymbol{\lambda}, \quad D_T = \frac{u_0^2 \tau_0}{3} . \tag{45}$$

Here we use the notation

$$\langle \mathbf{u}^2 \rangle = u_0^2 F_0(Z) . \tag{46}$$

Taking into account Eqs. (44) and (45) we can rewrite Eq. (43) for the mean number density of the particles in the form

$$\frac{\partial N}{\partial t} + \nabla \cdot [(\mathbf{v}_s - \boldsymbol{\lambda} D_T F_0(z))N] = \nabla \cdot [(D + D_T F_0(z))\nabla N] . \tag{47}$$

Equation (47) has an equilibrium solution:

$$\nabla N_0 = \left(\frac{\mathbf{v}_s}{D_T F_0(z)} - \boldsymbol{\lambda}\right) N_0 , \tag{48}$$

where N_0 is an equilibrium distribution of number density of particles. Hereafter we consider the case Pe $\gg 1$, i.e., $D_T \gg D$. Now we study stability of this equilibrium.

We seek for the solution of Eq. (47) in the form

$$N(t,\mathbf{r}) = N_0(\mathbf{r}) + N(t,Z)\exp(i\mathbf{k}\cdot\mathbf{r}_\perp) , \tag{49}$$

where the wave vector $\mathbf{k}$ is perpendicular to the axis Z. Substituting Eq. (49) into (47) yields

$$\frac{\partial N}{\partial t} = \frac{1}{m_0}\frac{\partial^2 N}{\partial Z^2} + \mu_0 \frac{\partial N}{\partial Z} - \frac{\kappa_0}{m_0} N , \tag{50}$$

where

$$\frac{1}{m_0} = F_0(Z), \quad \mu_0 = F_0' - \lambda F_0 - v_0,$$
$$\kappa_0 = k^2 - \lambda \frac{F_0'}{F_0} - \lambda' .$$

Hereafter we consider the case $F_0(Z) \gg$Pe^{-1} for all Z. Equation (50) is written in dimensionless form, coordinate Z is measured in units Λ_u, time t is measured in units Λ_u^2/D_T, the wave number k and value λ are measured in units Λ_u^{-1}, and $z = \Lambda_u Z$, $v_0 = v_s\Lambda_u/D_T \simeq 3m_p/(m_f\mathrm{Pe})$, and Λ_u is the characteristic scale of the spatial distribution $\langle \mathbf{u}^2\rangle$, the vector $\boldsymbol{\lambda} = \lambda \mathbf{e}_z$, $\mathbf{e}_z$ is the unit vector directed along the axis Z.

Substitution

$$N(t,Z) = \Psi_0(Z)\exp(\gamma_0 t)\exp\left[-\frac{1}{2}\int \chi_0\, dZ\right] \tag{51}$$

reduces Eq. (50) to the eigenvalue problem of the Schrödinger equation

$$\frac{1}{m_0}\Psi_0'' + [W_0 - U_0]\Psi_0 = 0 , \tag{52}$$

where $W_0 = -\gamma_0$, and the potential U_0 is given by

$$U_0 = \frac{1}{m_0}\left(\frac{\chi_0^2}{4} + \frac{\chi_0'}{2} + \kappa_0\right),$$
$$\chi_0 = \mu_0 m_0 = \frac{F_0'}{F_0} + \lambda - \frac{v_0}{F_0} . \tag{53}$$

Now we use a quantum mechanics analogy for the analysis of the pattern formation in a spatial distribution of the passive scalar. The instability ($\gamma_0 > 0$) can be excited if there is a region of potential well where $U_0 < 0$. The positive value of W_0 corresponds to the turbulent diffusion, whereas a negative value of W_0 results in the excitation of the instability. Next, we introduce a function $f = \ln\langle \mathbf{u}^2 \rangle$ and $f' = F_0'/F_0$. The potential U_0 can be rewritten as

$$U_0 = \frac{1}{m_0}\left(\frac{1}{4}(f' - \lambda)^2 + \frac{1}{2}[f'' - \lambda' - v_0 \lambda \exp(-f)] + k^2 + \frac{v_0^2}{4}\exp(-2f)\right) . \tag{54}$$

The potential U_0 can be negative in a region where $f'' - \lambda' - v_0 \lambda \exp(-f) < 0$. For instance, the function $f(Z)$ with a maximum can satisfy this condition. Next, we consider an example. We expand the functions $f(Z)$ and $\lambda(Z)$ in Taylor series at the point $Z = 0$, i.e., $f(Z) \sim -Z^2 + \cdots$ and $\lambda(Z) \sim \lambda_0 + a_0 Z + a_1 Z^3 + \cdots$. The point $Z = 0$ is at the maximum of the spatial distribution $\ln\langle \mathbf{u}^2 \rangle$, and the value $\ln\langle \mathbf{u}^2 \rangle$ is measured in units of value f at the point $Z = 0$. Note that $\boldsymbol{\nabla} \cdot \mathbf{u} = \boldsymbol{\lambda} \cdot \mathbf{u}$, and $\boldsymbol{\lambda} = -\boldsymbol{\nabla}\rho/\rho$. Strong inhomogeneity of the density ρ of the surrounding fluid can be caused for example by an inhomogeneity of the temperature of the surrounding fluid.

Expansion of the functions $f(Z)$ and $\lambda(Z)$ in Taylor series simplifies the Schrödinger equation (52) with variable mass. It is given by

$$\Psi_0'' + [W_0 - \tilde{U}(Z)]\Psi_0 = 0 , \tag{55}$$

where a potential $\tilde{U}(Z)$ is given by

$$\tilde{U}(Z) = A_0(Z + Z_*)^2 - U_* ,$$

with

$$A_0 = b^2 - \frac{3}{2}a_1 - \frac{v_0}{2}(\lambda_0 - v_0) - W_0, \quad Z_* = \frac{2b\lambda_0 - v_0 a_0}{4A_0},$$

$$U_* = b - \frac{1}{2}(\lambda_0 - v_0)^2 - k^2 + A_0 Z_*^2, \quad b = \frac{a_0}{2} + 1 .$$

Here $Z_* \ll 1$ and we can neglect the small term $A_0 Z_*^2$ in the expression for U_*.

Now we introduce a new variable

$$\zeta = A_0^{1/4}(Z + Z_*) , \tag{56}$$

where $A_0 > 0$. Then Eq. (55) reduces to

$$\frac{d^2\Psi_0}{d\zeta^2} + (E - \zeta^2)\Psi_0 = 0 , \tag{57}$$

where

$$E = A_0^{-1/2}(U_* + W_0) . \tag{58}$$

Equation (57) is similar to that for a harmonic oscillator in quantum mechanics (see, e.g., [47]). Discrete levels of energy of the oscillator are determined by conditions

$$E - 1 = 2p, \quad p = 0, 1, 2, \ldots . \tag{59}$$

Combination of Eqs. (59) and (58) yields the growth rate of the instability for $p = 0$

$$\gamma_0 = b + \frac{1}{2} - k^2 - \frac{1}{4}(\lambda_0 - v_0)^2 - \left[\left(b + \frac{1}{2}\right)^2 - \frac{3}{2}a_1 - k^2 - \frac{1}{4}(\lambda_0^2 - v_0^2)\right]^{\frac{1}{2}} .$$

Note that when $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, i.e., $\lambda = 0$ (for $\lambda_0 = a_0 = a_1 = 0$) the value $\gamma_0 < 0$ and the instability is not excited. Nonzero λ_0 reduces the effect of gravitation (sedimentation of particles). The growth rate of the instability has a maximum at $k = 0$. If v_0 is very small (when $\mathrm{Pe} \gg m_p/m_f$) the threshold of the instability is independent of the Péclet number Pe. It is determined by the value $\boldsymbol{\nabla} \cdot \mathbf{u}$ and the inhomogeneity of the turbulence. Note that the approximation of the harmonic quantum oscillator is valid when $\gamma_0 \sim U_*$. More detailed investigation of the instability is possible by means of numerical simulation.

Thus we have shown here that the initially spatial distribution of a number density of passive scalar particles evolves into a pattern containing regions with increased (decreased) concentration of a passive scalar. Characteristic vertical size of the inhomogeneity is of the order of $l_z \simeq A_0^{-1/4}\Lambda_u$ [see Eq. (56)]. Therefore the characteristic height of these "spots" is less than a scale Λ_u of inhomogeneity of turbulence. On the contrary, the characteristic horizontal size of these inhomogeneities is much greater than Λ_u, since the mode with maximum growth rate has a minimum lateral wave number k. Therefore the above analysis suggests formation of the panlike structures in inhomogeneous turbulent fluid flow with div $\mathbf{u} \neq 0$. A physics of this instability is considered in the next section.

VI. DISCUSSION

A theory of turbulent transport of a passive scalar (mean field and fluctuations) in random compressible flow is developed. In this investigation we suggested a possible mechanism of slow (inhibited) diffusion of small-scale fluctuations of a passive scalar in a compressible (i.e., div $\mathbf{u} \neq 0$) turbulent velocity field. The reason for slow diffusion is that accumulation and outflow of the particles of the passive scalar in a small volume are separated in time and are not balanced in a compressible flow. Molecular diffusion breaks a symmetry between accumulation and outflow, i.e., it breaks a reversibility in time and does not allow leveling of the total mass fluxes over the consecutive intervals of time. Certainly the compressibility of the turbulent flow results only in a redistribution of the particles of the passive scalar in the volume. In the whole volume the total quantity of particles of the passive scalar is conserved.

It is shown also that the magnitude of the fluctuations of the passive scalar generated in the presence of external gradient of the mean mass concentration $\boldsymbol{\nabla} Q$ in compressible flow of fluid can be fairly strong: $\sqrt{\langle q^2 \rangle} \sim l_0 \ln(\mathrm{Pe})|\boldsymbol{\nabla} Q|$, where l_0 is the characteristic scale of the turbulent velocity field.

In Sec. IV in order to study passive scalar fluctuations we considered a model of a homogeneous and isotropic turbulence with $\langle(\boldsymbol{\nabla} \cdot \mathbf{u})^2\rangle \neq 0$. Generally the velocity $\mathbf{u}$ of such a flow has potential and vortical components. The simple model of the turbulent velocity field analyzed in this work was chosen for simplicity and to make the calculations more transparent for the reader. Also it is done to elucidate a mechanism of slow diffusion of fluctuations of the passive scalar concentration in a compressible flow. The obtained results are likely to be valid also in more sophisticated and realistic models of compressible turbulent flow with div $\mathbf{u} \neq 0$. The suggested mechanism of slow diffusion of passive scalar fluctuations can be of relevance in atmospheric flows and turbulent combustion.

It was also demonstrated in this study that the initially large-scale distribution of a passive scalar in a compressible turbulence ($\nabla \cdot \mathbf{u} \neq 0$) evolves generally into a strongly inhomogeneous large-scale structure. This pattern formation in the passive scalar field is caused by the instability which can be excited under certain conditions. Remarkably this effect disappears in the incompressible case, i.e., when $\nabla \cdot \mathbf{u} = 0$.

This instability is related to the appearance of an additional flux of particles. This flux of particles cannot be reduced either to turbulent diffusion or to flux of particles due to mean flow. Now we calculate the additional flux of particles. Equation (1) for the number density of particles for large Péclet numbers yields

$$\nabla \cdot \mathbf{v}_p \propto -\frac{1}{n_p}\frac{dn_p}{dt}.$$

On the other hand, a nonzero $\nabla \cdot \mathbf{v}_p$ for the particles is determined by the value $\nabla \cdot \mathbf{v}$ for the surrounding fluid, because $\mathbf{v}_p = \mathbf{v} + \mathbf{v}_s$ and $\nabla \cdot \mathbf{v}_s = 0$ (see Sec. V A). Therefore the change of number density of the particles δn_p for the turnover time τ_0 of turbulent eddies is given by $\delta n_p \propto -n_p \tau_0 \nabla \cdot \mathbf{v}$. The mean flux of particles is given by

$$\mathbf{J}_p = \langle \mathbf{u} \delta n_p \rangle \propto -\tau_0 N \langle \mathbf{u}(\nabla \cdot \mathbf{u}) \rangle \ ,$$

where $\mathbf{u}$ is the turbulent velocity of the surrounding fluid. In particular, when $\nabla \cdot \mathbf{u} = \boldsymbol{\lambda} \cdot \mathbf{u}$ we obtain

$$\mathbf{J}_p \propto -\tau_0 N \langle \mathbf{u}^2 \rangle \boldsymbol{\lambda} \ .$$

Evolution of the mean number density of the passive scalar particles is determined by

$$\frac{\partial N}{\partial t} + \nabla \cdot \mathbf{J}_p = -\nabla \cdot \mathbf{J}_T \ ,$$

where $\mathbf{J}_T = -\hat{D} \nabla N$ is the flux of the passive scalar particles caused by turbulent diffusion. Therefore

$$\frac{\partial N}{\partial t} \propto \nabla \cdot (\tau_0 \langle \mathbf{u}^2 \rangle \boldsymbol{\lambda} N) - \nabla \cdot \mathbf{J}_T \ .$$

Thus, formation of inhomogeneous large-scale structures of the passive scalar particles is caused by compressible ($\nabla \cdot \mathbf{u} \neq 0$) and inhomogeneous fluid flow. This pattern formation is not accompanied by an accumulation of mass of the surrounding fluid [see Sec. V C].

The large-scale pattern formation can occur only in an inhomogeneous turbulent flow since the compressibility alone is not sufficient to excite the instability. The inhomogeneous turbulence occurs in various flows, e.g., turbulent boundary layer or turbulent stratified flows. Notably the characteristic size of a large-scale pattern is less than the characteristic length scale Λ_u of $\langle \mathbf{u}^2 \rangle$ variation.

TABLE I. The parameters for aerosol particles in the atmospheric turbulent boundary layer.

a_* (μm)	D (cm^2/s)	Pe	v_0/ρ_m (cm^3/g)
0.1	2.2×10^{-6}	$(1.3 - 45) \times 10^{10}$	$(5.6 - 190) \times 10^{-4}$
1	1.3×10^{-7}	$(2.3 - 77) \times 10^{11}$	$(3.2 - 110) \times 10^{-2}$
10	1.4×10^{-8}	$(2.1 - 71) \times 10^{12}$	$3.5 - 120$

The analyzed instability may be of relevance in some atmospheric phenomena, e.g., atmospheric aerosols (see, e.g., [50–52]). Now we consider the latter phenomenon in more detail. The atmospheric turbulence exists in a layer that is located at an altitude from 100 to 1500 m from the Earth's surface. In this region an atmospheric turbulent boundary layer is formed. The characteristic values of parameters in this layer are as follows: $u_0 \sim 30 - 100$ cm/s; $l_0 \sim 10^3 - 10^4$ cm. The coefficient of diffusion D, the Péclet number Pe $= u_0 l_0 / D$, and the value $v_0/\rho_m = 4\pi a_*^3/(29 m_+ \mathrm{Pe})$ depend on the size of the aerosol particles a_*. These parameters are presented in Table I for different a_* (see, e.g., [50–52]). Here ρ_m is the density of the matter of an aerosol particle, m_+ is the mass of proton. The typical value of the density $\rho_m \sim 2$ g/cm^3. The value Λ_u can be estimated as the characteristic size of the turbulent boundary layer, i.e., $\Lambda_u \sim (1 - 15) \times 10^4$ cm. The latter implies that the characteristic time of excitation of the instability $\tau \sim \gamma_0^{-1} \sim \Lambda_u^2 / D_T$ varies from 1 to 10 h. The value Pe $\sim D^{-1} \sim a_*$ and the damping rate of the instability due to the gravitation (i.e., sedimentation of the particles) is of the order of $v_0 \sim a_*^2$. Using this and the data in Table I we can estimate a maximum size of the particles for which the maximum $\gamma_0 > 0$. The result is given by $a_* < 4 - 30$ μm. Note in passing that the above condition on the particle size in the inhomogeneous patterns is in compliance with the measured size of particles in regions with high concentrations of atmospheric aerosols.

The maximum value of the concentration of the particles of the passive scalar (or aerosol particles) inside the inhomogeneity formed due to the excitation of instability can be found by means of the nonlinear theory. One of the possible nonlinear mechanisms of stabilization of the instability is related to a dynamic coupling of particles and surrounding turbulent fluid flow.

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APPENDIX A: SOLUTION OF EQ. (3)

Let us check that Eq. (5) is a solution of Eq. (3). According to Eq. (5) in a short time Δt the scalar field varies as

$$C(t+\Delta t, \mathbf{x}) = M\{C(t, \boldsymbol{\xi}_{\Delta t})\} .$$

We expand the function $C(t, \boldsymbol{\xi}_{\Delta t})$ in the Taylor series in the vicinity of the point $\mathbf{x}$:

$$C(t, \boldsymbol{\xi}_{\Delta t}) \simeq C(t, \mathbf{x}) + \frac{\partial C}{\partial x_m}(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m + \frac{1}{2}\frac{\partial^2 C}{\partial x_m \partial x_p}(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_p + \cdots . \tag{A1}$$

It follows from Eq. (4) for the Wiener trajectory that

$$(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m = -\int_0^{\Delta t} v_m(t_s, \boldsymbol{\xi}_s)\, ds + (2D)^{1/2}\mathbf{w}_m(\Delta t) . \tag{A2}$$

Expanding the velocity $v_m(t_s, \boldsymbol{\xi}_s)$ in the Taylor series in the vicinity of the point $\mathbf{x}$, and using Eq. (A2), yields

$$v_m(t_s, \boldsymbol{\xi}_s) = v_m(t_s, \mathbf{x}) + \frac{\partial v_m}{\partial x_l}\left[-\int_0^s v_l(t_\sigma, \mathbf{x})\, d\sigma + (2D)^{1/2}\mathbf{w}_l(s)\right] . \tag{A3}$$

Substitution of (A3) into (A2) yields

$$(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m = -\int_0^{\Delta t} v_m(t_s, \boldsymbol{\xi}_s)\, ds + \int_0^{\Delta t} \left.\frac{\partial v_m}{\partial x_l}\right|_{(t_s, \mathbf{x})} ds \int_0^s v_l(t_\sigma, \boldsymbol{\xi}_\sigma)\, d\sigma - \sqrt{2D}\int_0^{\Delta t} \left.\frac{\partial v_m}{\partial x_l}\right|_{(t_s, \mathbf{x})} w_l(s)\, ds + \sqrt{2D} w_m . \tag{A4}$$

We calculate the integrals in (A4) by means of the "mean value" theorem. The result is given by

$$(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m = -v_m(t_1, \mathbf{x})\Delta t + \sqrt{2D} w_m + O[(\Delta t)^2] , \tag{A5}$$

where t_1 is within the interval $(t, t+\Delta t)$. Substitution of Eq. (A5) into (A1) and averaging over Wiener trajectories yields the field $C(t+\Delta t, \mathbf{x})$. Now we calculate the value $[C(t+\Delta t, \mathbf{x}) - C(t, \mathbf{x})]/\Delta t$ for $\Delta t \to 0$. This procedure results in the equation

$$\frac{\partial C}{\partial t} = -v_m \frac{\partial C}{\partial x_m} + \frac{1}{2}\frac{\partial^2 C}{\partial x_m \partial x_p}(2D\delta_{mp}) . \tag{A6}$$

Equation (A6) coincides with Eq. (3). Therefore Eq. (5) is a solution of Eq. (3).

APPENDIX B: EQUATION FOR ΔC

The total field $C(t+\Delta t)$ at instant $t+\Delta t$ is expressed in terms of the field $C(t)$ by means of the equation

$$C(t+\Delta t, \mathbf{x}) = M\{C(t, \boldsymbol{\xi}_{\Delta t})\} . \tag{B1}$$

We expand the function $C(t, \boldsymbol{\xi}_{\Delta t})$ in the Taylor series in the vicinity of the point $\mathbf{x}$:

$$C(t, \boldsymbol{\xi}_{\Delta t}) \simeq C(t, \mathbf{x}) + \frac{\partial C}{\partial x_m}(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m + \frac{1}{2}\frac{\partial^2 C}{\partial x_m \partial x_p}(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_p + \cdots . \tag{B2}$$

Using Eq. (4) for the Wiener trajectory we obtain

$$(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m = -\int_0^{\Delta t} v_m(t_s, \boldsymbol{\xi}_s)\, ds + (2D)^{1/2}\mathbf{w}_m(\Delta t) . \tag{B3}$$

Expanding the velocity $v_m(t_s, \boldsymbol{\xi}_s)$ in the Taylor series in the vicinity of the point $\mathbf{x}$, and using Eq. (B3), yields

$$v_m(t_s, \boldsymbol{\xi}_s) \simeq v_m(t_s, \mathbf{x}) - v_l \frac{\partial v_m}{\partial x_l} s + (2D)^{1/2}\frac{\partial v_m}{\partial x_l}\mathbf{w}_l(s) + \cdots . \tag{B4}$$

It is assumed that the velocity $\mathbf{v}$ is constant (time independent) in small intervals $(0, \Delta t); (\Delta t, 2\Delta t); \ldots,$ and changes every small time Δt (the velocity is statistically independent in the different time intervals). Substitution of (B4) into (B3) and calculation of the integrals in (B3) accurate up to $\sim (\Delta t)^2$ yield

$$(\boldsymbol{\xi}_{\Delta t} - \mathbf{x})_m \simeq -v_m \Delta t + \frac{1}{2} v_l \frac{\partial v_m}{\partial x_l}(\Delta t)^2 - \sqrt{2D}\frac{\partial v_m}{\partial x_l}\int_0^{\Delta t} w_l\, ds + \sqrt{2D} w_m + \cdots . \tag{B5}$$

Combination of Eqs. (B5), (B2), and (B1) yields

$$C(t+\Delta t,\mathbf{x}) = M\Bigg\{C(t,\mathbf{x}) + \frac{\partial C}{\partial x_m}\bigg(-v_m\Delta t + \frac{1}{2}v_l\frac{\partial v_m}{\partial x_l}(\Delta t)^2 + \sqrt{2D}w_m - \sqrt{2D}\frac{\partial v_m}{\partial x_l}\int_0^{\Delta t} w_l\, ds\bigg) + \frac{1}{2}\frac{\partial^2 C}{\partial x_m \partial x_p}\Big[v_m v_p(\Delta t)^2 + 2Dw_m w_p - \sqrt{2D}\Delta t(v_m w_p + v_p w_m)\Big]\Bigg\} ,$$

where we keep terms $\geq O[(\Delta t)^2]$.

APPENDIX C: EQUATION FOR THE MEAN PASSIVE SCALAR FIELD FOR SMALL PÉCLET NUMBERS

The solution of the equation of convective diffusion (3) in the form (5) by averaging over the Wiener trajectories is valid for arbitrary Péclet numbers. However, using a δ-correlated in time process for the turbulent velocity field is possible only for $\mathrm{Pe} \gg 1$. In this appendix we will show that for small Péclet numbers the equation for the mean passive scalar field has the same form as in the case $\mathrm{Pe} \gg 1$.

The equation of the convective diffusion (3) can be rewritten in the form

$$\frac{\partial C}{\partial t} + \boldsymbol{\nabla}\cdot(C\mathbf{v}) = D\Delta C + C\boldsymbol{\nabla}\cdot\mathbf{v} . \tag{C1}$$

The total fields $\mathbf{v}$ and C can be presented in the form $\mathbf{v} = \mathbf{V} + \mathbf{u}$ and $C = Q + q$ where $\mathbf{V} = \langle \mathbf{v}\rangle$ and $Q = \langle C\rangle$, q is a turbulent component of the mass concentration of a passive scalar, and the angular brackets mean statistical averaging. Averaging Eq. (C1) over the ensemble of the turbulent pulsations we obtain the equation for the mean field Q

$$\frac{\partial Q}{\partial t} + (\mathbf{V}\cdot\boldsymbol{\nabla})Q - D\Delta Q = \langle q\boldsymbol{\nabla}\cdot\mathbf{u}\rangle - \boldsymbol{\nabla}\cdot\langle q\mathbf{u}\rangle . \tag{C2}$$

Subtraction of Eq. (C2) from (C1) yields the equation for the turbulent field q

$$\frac{\partial q}{\partial t} - D\Delta q = -(\mathbf{u}\cdot\boldsymbol{\nabla})Q . \tag{C3}$$

Equation (C3) is written in a frame moving with the mean velocity $\mathbf{V}$. Here we neglect the small quadratic in the fluctuating field terms $(\mathbf{u}\cdot\boldsymbol{\nabla})q - \langle(\mathbf{u}\cdot\boldsymbol{\nabla})q\rangle$. These terms yield effects that are of the order of $\sim \mathrm{Pe}^2$, whereas linear in the fluctuating field terms are of the order of $\sim \mathrm{Pe}$.

A solution of Eq. (C3) with the initial condition $q(t=0,\mathbf{x}) = q_0(\mathbf{x})$ is given by

$$q(t,\mathbf{x}) = \int q_0(\mathbf{z})G(t,\mathbf{x}-\mathbf{z})\, d^3z - \int u_m(t',\mathbf{z})\frac{\partial Q}{\partial z_m}G(t-t',\mathbf{x}-\mathbf{z})\, d^3z\, dt' , \tag{C4}$$

where $G(\tau,\mathbf{y})$ is the diffusion Green's function

$$G(\tau,\mathbf{y}) = (2\pi D\tau)^{-3/2}\exp\left(-\frac{y^2}{2\pi D\tau}\right) .$$

Now let us calculate the second moment $\langle q u_n\rangle$ by means of Eq. (C4). The result is given by

$$\langle q(t,\mathbf{y})u_n(t,\mathbf{x})\rangle = \int \langle q_0(\mathbf{z})u_n(t,\mathbf{x})\rangle G(t,\mathbf{y}-\mathbf{z})\, d^3z - \int \langle u_n(t,\mathbf{x})u_m(t',\mathbf{z})\rangle \frac{\partial Q}{\partial z_m}(t',\mathbf{z}) \times G(t-t',\mathbf{y}-\mathbf{z})\, d^3z\, dt' . \tag{C5}$$

Note that $\langle q_0 u_n\rangle = 0$, because q_0 and $\mathbf{u}$ are not correlated. Next, we introduce the fast $\mathbf{r} = \mathbf{x} - \mathbf{z}$ and slow $\mathbf{R} = (\mathbf{x}+\mathbf{z})/2$ variables. The derivative

$$\frac{\partial Q}{\partial z_m} \simeq \frac{\partial Q}{\partial R_m} + O\left(\frac{r}{R}\right) .$$

It follows from Eq. (C5) that

$$\langle q(t,\mathbf{y})u_n(t,\mathbf{x})\rangle = -\frac{\partial Q}{\partial R_m}\tilde{D}_{mn} , \tag{C6}$$

where

$$\tilde{D}_{mn} = \int \langle u_n u_m\rangle G(\tau,\mathbf{r})\, d^3r\, d\tau .$$

Similar calculations for $\langle q\boldsymbol{\nabla}\cdot\mathbf{u}\rangle$ yield

$$\langle q\boldsymbol{\nabla}\cdot\mathbf{u}\rangle = -(\tilde{\mathbf{V}}_{\text{eff}}\cdot\boldsymbol{\nabla})Q , \tag{C7}$$

where

$$\tilde{\mathbf{V}}_{\text{eff}} = \int \langle \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{u})\rangle G(\tau,\mathbf{r})\, d^3r\, d\tau .$$

Substitution of (C6) and (C7) into Eq. (C2) yields the equation for the mean passive scalar field

$$\frac{\partial Q}{\partial t} + (\mathbf{V}_{\text{eff}}\cdot\boldsymbol{\nabla})Q = \frac{\partial}{\partial R_m}\left(D_{mn}\frac{\partial Q}{\partial R_n}\right) , \tag{C8}$$

where

$$D_{mn} = D\delta_{mn} + \int \langle u_m u_n\rangle G(\tau,\mathbf{r})\, d^3r\, d\tau , \tag{C9}$$

$$\mathbf{V}_{\text{eff}} = \mathbf{V} + \int \langle \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{u})\rangle G(\tau,\mathbf{r})\, d^3r\, d\tau . \tag{C10}$$

Comparison of Eqs. (C8)–(C10) obtained for $\mathrm{Pe} \ll 1$ with Eqs. (7)–(9) derived for $\mathrm{Pe} \gg 1$ shows that these equations coincide in form. Therefore the above described phenomena for $\mathrm{Pe} \gg 1$ can also occur for $\mathrm{Pe} \ll 1$. However, the instability which can be excited at $\mathrm{Pe} \gg 1$ is suppressed for $\mathrm{Pe} \ll 1$ by strong molecular diffusion and sedimentation.

APPENDIX D: MODEL OF A COMPRESSIBLE HOMOGENEOUS AND ISOTROPIC TURBULENT VELOCITY FIELD

We consider a model of the turbulent velocity field $\mathbf{u}$ with $\nabla\cdot\mathbf{u} \neq 0$. This model corresponds to a homogeneous and isotropic turbulence with small, but finite value of $\langle(\nabla\cdot\mathbf{u})^2\rangle$. We present the velocity $\mathbf{u}$ as a sum of two components: the potential component $\mathbf{u}^{(P)} = \nabla P$ and the vortex component $\mathbf{u}^{(\Omega)} = \nabla\times\mathbf{\Omega}$. In $\mathbf{k}$ space the second moment is given by

$$\langle u_m(\mathbf{k}_1)u_n(\mathbf{k}_2)\rangle = \langle u_m^{(P)}u_n^{(P)}\rangle + \langle u_m^{(P)}u_n^{(\Omega)}\rangle + \langle u_m^{(\Omega)}u_n^{(P)}\rangle + \langle u_m^{(\Omega)}u_n^{(\Omega)}\rangle \ . \qquad \text{(D1)}$$

All these tensors are proportional to $\sim \delta(\mathbf{k}_1+\mathbf{k}_2)$ since the turbulence is assumed to be homogeneous. Note that in isotropic turbulence

$$\langle u_m^{(P)}u_n^{(\Omega)}\rangle = \langle u_m^{(\Omega)}u_n^{(P)}\rangle = 0 \ .$$

Indeed, in $\mathbf{k}$ space $u_m^{(P)} = iP(k)k_m$ and $u_n^{(\Omega)} = i\varepsilon_{nls}k_l\Omega_s$, where ε_{nls} is the tensor of Levi-Cività. Therefore

$$\langle u_m^{(P)}u_n^{(\Omega)}\rangle = k_m k_l \varepsilon_{nls}\langle P(k)\Omega_s\rangle = 0 \ ,$$

since $\langle P(k)\Omega_s\rangle = 0$ in isotropic turbulence. The tensor

$$\langle u_m^{(\Omega)}u_n^{(\Omega)}\rangle = \frac{f_\Omega(k)}{4\pi k^2}(k^2\delta_{mn} - k_m k_n) \qquad \text{(D2)}$$

(see, e.g., [7,27]). On the other hand,

$$\langle u_m^{(P)}u_n^{(P)}\rangle = \frac{f_P(k)}{4\pi k^2}k_m k_n \ . \qquad \text{(D3)}$$

Now taking into account (D1)–(D3) and returning to $\mathbf{r}$ space we obtain

$$\langle u_m(\mathbf{x})u_n(\mathbf{y})\rangle = -\left(\delta_{mn}\Delta - \frac{\partial^2}{\partial r_m\partial r_n}\right)f_\Omega(\mathbf{r}) - \frac{\partial^2}{\partial r_m\partial r_n}f_P(\mathbf{r}) \ , \qquad \text{(D4)}$$

where we use Eqs. (14) and (15) and substitutions $k^2 \to -\Delta$ and $ik_m \to \partial/\partial r_m$. Now we introduce the following functions:

$$F(r) = -\frac{1}{r^3}\frac{\partial}{\partial r}\left[r^3\frac{\partial}{\partial r}f_\Omega(\mathbf{r})\right], \quad F_c(r) = -\frac{1}{r}\frac{\partial}{\partial r}f_P(\mathbf{r}) \ .$$

Using Eqs. (14) and (D4) after simple calculations we obtain the correlation function of the velocity field

$$\langle u_m(\mathbf{x})u_n(\mathbf{y})\rangle = \frac{u_0^2}{3}\left[[F(r)+F_c(r)]\delta_{mn} + \frac{r}{2}\frac{dF}{dr}\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right) + r\frac{dF_c}{dr}\frac{r_m r_n}{r^2}\right] .$$

Now we calculate the value $\langle(\nabla\cdot\mathbf{u})^2\rangle$

$$\langle(\nabla\cdot\mathbf{u})^2\rangle \equiv -\frac{\partial^2}{\partial r_m\partial r_n}\langle u_m u_n\rangle\Big|_{r\to 0} = -\frac{8}{3}u_0^2\left(\frac{F_c'}{r} + \frac{7}{8}F_c''\right)_{r\to 0} .$$

For $r \ll 1$ the function $F_c = \varepsilon(1-\alpha r^2)$ and

$$\langle(\nabla\cdot\mathbf{u})^2\rangle = 10\varepsilon\alpha u_0^2 \ .$$

APPENDIX E: MODEL OF AN INHOMOGENEOUS TURBULENT VELOCITY FIELD IN A STRATIFIED MEDIUM

Consider a model of the turbulent velocity field $\mathbf{u}$ that corresponds to the flow in a stratified medium. In this case $\nabla\cdot\mathbf{u} \neq 0$. The stratification is caused by inhomogeneous stationary distribution of the density. Then the continuity equation $\nabla\cdot(\rho\mathbf{u}) = 0$ can be rewritten in the form $\nabla\cdot\mathbf{u} = (\boldsymbol{\lambda}\cdot\mathbf{u})$ where $\boldsymbol{\lambda} = -\nabla\rho/\rho$. In $\mathbf{k}$ space it is given by

$$(\mathbf{k}\cdot\mathbf{u}) = -i(\boldsymbol{\lambda}\cdot\mathbf{u}) \ . \qquad \text{(E1)}$$

Consider a tensor $\langle u_m(\mathbf{y})u_n(\mathbf{x})\rangle$ assuming that this tensor in $\mathbf{r}$ space depends only on $\mathbf{r} = \mathbf{x} - \mathbf{z}$ and it is independent of $\mathbf{R} = (\mathbf{x}+\mathbf{z})/2$. This is valid for the case $\Lambda_u \gg \Lambda$, where Λ_u is the characteristic scale of the $\langle u^2\rangle$ variation. Thus the tensor $\langle u_m(\mathbf{y})u_n(\mathbf{x})\rangle$ in $\mathbf{k}$ space is

$$\langle u_m(\mathbf{k}_1)u_n(\mathbf{k}_2)\rangle \sim \delta(\mathbf{k}_1+\mathbf{k}_1) \ .$$

Therefore $\mathbf{k}_1 = -\mathbf{k}_2 \equiv \mathbf{k}$. A general reflectively invariant form of the tensor $\langle u_m(\mathbf{k})u_n(-\mathbf{k})\rangle$ is given by

$$f_{mn}(\mathbf{k}) \equiv \langle u_m(\mathbf{k})u_n(-\mathbf{k})\rangle = B_1\delta_{mn} + B_2k_mk_n + B_3\lambda_m\lambda_n + B_4k_m\lambda_n + B_5k_n\lambda_m \ . \qquad \text{(E2)}$$

Note that the results also can be valid for a weak dependence of the tensor on $\mathbf{R}$. In this case the coefficients B_j slowly vary with $\mathbf{R}$. The tensor $f_{mn}(\mathbf{k})$ must satisfy the following identities:

$$f_{mn}(\mathbf{k}) = f_{nm}(-\mathbf{k}) \ , \qquad \text{(E3)}$$

$$f_{mn}(-\mathbf{k}) = f_{mn}^*(\mathbf{k}) \ . \qquad \text{(E4)}$$

The condition (E3) yields $B_4 = -B_5$ and (E4) yields $B_4 = iB_0$. It follows from (E1) that

$$f_{mn}k_n = -if_{mn}\lambda_n \ . \qquad \text{(E5)}$$

Equation (E5) yields $B_1 = B_0(k^2+\lambda^2)$ and $B_2 = B_3 = -B_0$. Therefore the tensor (E5) is given by

$$f_{mn}(\mathbf{k}) \equiv \langle u_m(\mathbf{k})u_n(-\mathbf{k})\rangle = B_0[(k^2+\lambda^2)\delta_{mn} - k_mk_n - \lambda_m\lambda_n + i(k_m\lambda_n - k_n\lambda_m)] \ . \qquad \text{(E6)}$$

We assume for simplicity that B_0 depends only on $|\mathbf{k}|$. In $\mathbf{r}$ space the tensor (E6) is given by

$$\langle u_m(\mathbf{y})u_n(\mathbf{x})\rangle = B_*\biggl[F(r)\delta_{mn} + \frac{r}{2}\frac{dF}{dr}\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right) - \frac{1}{2}F(r)(r_m\lambda_n - r_n\lambda_m) + \lambda^2 A(r)\left(\delta_{mn} - \frac{\lambda_m\lambda_n}{\lambda^2}\right)\biggr] \quad \text{(E7)}$$

where we use notations

$$\langle u_m(\mathbf{y})u_n(\mathbf{x})\rangle = \int \langle u_m(\mathbf{k})u_n(-\mathbf{k})\rangle \exp(-i\mathbf{k}\cdot\mathbf{r})\, d^3k, \qquad ik_n \to \frac{\partial}{\partial r_n}\,, \quad \text{(E8)}$$

and

$$\langle u^2\rangle = \langle u_n u_n\rangle, \quad F(r) = -\frac{2}{r}\frac{dA}{dr}\,. \quad \text{(E9)}$$

Expression (E7) is valid for the case $\Lambda_u \gg \Lambda$. Now we consider a general case of arbitrary Λ_u and Λ, i.e., an inhomogeneous turbulence. In this case the tensor $\langle u_m(\mathbf{y})u_n(\mathbf{x})\rangle$ in $\mathbf{r}$ space depends on $\mathbf{r}$ and on $\mathbf{R}$. In this case the correlation function $\langle u_m u_n\rangle$

$$\langle u_m(\mathbf{y})u_n(\mathbf{x})\rangle = \int \langle u_m(\mathbf{k}_1)u_n(\mathbf{k}_2)\rangle \times \exp i(\mathbf{k}_1\cdot\mathbf{x} + \mathbf{k}_2\cdot\mathbf{y})\, d\mathbf{k}_1\, d\mathbf{k}_2 = \int \tilde{\Phi}_{mn}(\mathbf{K},\mathbf{r})\exp(i\mathbf{K}\cdot\mathbf{R})\, d\mathbf{K}\,,$$

where

$$\tilde{\Phi}_{mn}(\mathbf{K},\mathbf{r}) = \int \langle u_m(-\mathbf{k}+\mathbf{K}/2)u_n(\mathbf{k}+\mathbf{K}/2)\rangle \times \exp(i\mathbf{k}\cdot\mathbf{r})\, d\mathbf{k},$$

$$\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2, \quad \mathbf{k} = \frac{1}{2}(\mathbf{k}_2 - \mathbf{k}_1)\,,$$

and $\mathbf{R}$ and $\mathbf{K}$ correspond to the large scales, and $\mathbf{r}$ and $\mathbf{k}$ to the small ones (see, e.g., [53,54]).

For an inhomogeneous turbulence the continuity equation $\boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0$ is written in the form

$$f_{mn}\left(k_n + \frac{K_n}{2}\right) = -if_{mn}\lambda_n\,. \quad \text{(E10)}$$

Equation (E10) coincides with Eq. (E5) for $K_n = 0$, i.e., for homogeneous turbulence. Equation (E10) can be rewritten in the form

$$f_{mn}k_n = -if_{mn}\tilde{\lambda}_n\,, \quad \text{(E11)}$$

where

$$\tilde{\lambda}_n = \lambda_n - \frac{iK_n}{2}\,. \quad \text{(E12)}$$

Comparison of (E11) with (E5) shows that a substitution of (E12) instead of λ_n in the tensor (E7) yields a general form of the tensor $\langle u_m(\mathbf{k}_1)u_n(\mathbf{k}_2)\rangle$ for an inhomogeneous turbulence. The result is given by

$$f_{mn}(\mathbf{k},\mathbf{K}) \equiv \langle u_m(\mathbf{k}_1)u_n(\mathbf{k}_2)\rangle = B_0(k,\mathbf{K})\biggl[\left(k^2 + \lambda^2 - i(\boldsymbol{\lambda}\cdot\mathbf{K}) - \frac{1}{4}K^2\right)\delta_{mn} - k_m k_n - \lambda_m\lambda_n + i(k_m\lambda_n - k_n\lambda_m) + \frac{1}{4}K_m K_n + \frac{i}{2}(K_m\lambda_n + K_n\lambda_m) + \frac{1}{2}(k_m K_n - k_n K_m)\biggr]\,. \quad \text{(E13)}$$

To obtain this tensor in $\mathbf{r}$ space we have to replace

$$ik_n \to \frac{\partial}{\partial r_n}, \quad iK_n \to \frac{\partial}{\partial R_n}$$

in (E13). This procedure is equivalent to a replacement

$$\lambda_n \to \lambda_n - \frac{1}{2}\frac{\partial}{\partial R_n}$$

in the tensor (E7). The result is given by

$$\langle u_m(\mathbf{y})u_n(\mathbf{x})\rangle = B_*\biggl[F_0(r,\mathbf{R})\delta_{mn} + \frac{r}{2}\frac{\partial F_0}{\partial r}\left(\delta_{mn} - \frac{r_m r_n}{r^2}\right) - \frac{1}{2}F_0(r,\mathbf{R})(r_m\lambda_n - r_n\lambda_m) + \lambda^2 A(r,\mathbf{R})\left(\delta_{mn} - \frac{\lambda_m\lambda_n}{\lambda^2}\right) - \frac{1}{4}\left(r_m\frac{\partial}{\partial R_n} - r_n\frac{\partial}{\partial R_m}\right)F_0(r,\mathbf{R}) + \frac{1}{4}\delta_{mn}\left(\frac{\partial^2}{\partial R^2} - 4\lambda_p\frac{\partial}{\partial R_p}\right)A(r,\mathbf{R}) - \frac{1}{4}\frac{\partial}{\partial R_m}\frac{\partial}{\partial R_n}A(r,\mathbf{R}) + \frac{1}{2}\left(\lambda_m\frac{\partial}{\partial R_n} + \lambda_n\frac{\partial}{\partial R_m}\right)A(r,\mathbf{R})\biggr]\,. \quad \text{(E14)}$$

Note that the condition (E3) in inhomogeneous turbulence in $\mathbf{r}$ space is given by

$$f_{mn}(\mathbf{r},\mathbf{R}) = f_{nm}(-\mathbf{r},\mathbf{R}) \,.$$

The tensor (E14) satisfies this condition.

In the case when the vector $\partial F_0/\partial \mathbf{R}$ as well as $\partial A/\partial \mathbf{R}$ are directed along $\boldsymbol{\lambda}$ and for $\mathbf{x}=\mathbf{y}=\mathbf{z}$ the tensor (E14) is given by

$$\langle u_m(\mathbf{z})u_n(\mathbf{z})\rangle = \frac{\langle u^2\rangle}{3}\left(1+\frac{l_0^2\lambda^2}{6}\alpha(\mathbf{z})\right)^{-1}\left[\delta_{mn}+\frac{l_0^2\lambda^2}{4}\alpha(\mathbf{z})\left(\delta_{mn}-\frac{\lambda_m\lambda_n}{\lambda^2}\right)\right],$$

where

$$\alpha(\mathbf{z}) = 1+\frac{1}{4\lambda^2}\frac{F_0''}{F_0}-\frac{1}{4\lambda}\frac{F_0'}{F_0} \,.$$

Here we use the relations valid for $\mathbf{r}\to 0$

$$F_0(r,\mathbf{R}) = F_0(\mathbf{R})\left(1-\frac{r^2}{l_0^2}\right), \quad A(r,\mathbf{R}) = A(\mathbf{R})\left(1-\frac{r^2}{l_0^2}\right) .$$

Therefore [see Eq. (E9)]

$$F_0(\mathbf{R}) = \frac{4}{l_0^2}A(\mathbf{R}) \,.$$

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